AN ALGEBRAIC TREATMENT OF THE THEOREM OF CLOSURE.

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Introduction.

1. Among the classic theorems concerning the projective properties of a pair of conics, perhaps the most interesting is one due to Poncelet, viz. the theorem that if a polygon of n sides can be circumscribed about one conic and at the same time inscribed in a second conic, it is possible to construct an infinite number of such polygons for the given pair of conics. A very elegant demonstration of this theorem may be made by the use of elliptic functions, but a parallel algebraic treatment is also possible. From an algebraic point of view, we have here but one example of a certain interesting class of problems in elimination. We shall mention the general algebraic problem, but shall carry through the details only in the hyperelliptic case. Except in so far as is necessary to make the algebraic steps clear no discussion will be made of the numerous geometric corollaries that suggest themselves.

The present treatment is an attempt to reduce the problem to its simplest form and to prove the theorems needed with a minimum of algebraic machinery. Little emphasis is placed upon the numerous features which serve to individualize the elliptic within the general hyperelliptic problem.

The functions considered are those well-known in the transcendental theory, although the methods of proof are of necessity largely new. Constant use has been made of the remarkably clearly written Traité des Fonctions Elliptiques by Halphen.

It should be noted that not only are the operations used in this paper algebraic, but that except for a single irrationality, $\sqrt{\Delta}$, every step is essentially rational. Neither the notions of geometric continuity nor of convergence of series are required at any stage. Thus the present discussion is applicable in its entirety to finite fields, a statement which does not hold true of the algebraic treatments already published.

Extensive references to the literature on the problem of closure in the elliptic case may be found in the Encyklopädie der Math. Wiss., III, C 1, p. 45 ff., the Encyk. der Geometrie (Simon), p. 105 ff. and in Pascal's Repertorium, II[I], p. 238 ff.

Modern algebraic treatments of the Poncelet Polygons are given by

K. Petr, "Ponceletsche Polygone," Monatsh. f. Math. u. Phys. 18, (1907), pp. 122–137, and Karl Rohn, "Das Schliessungsproblem von Poncelet . . . ," Berichte d. Ges. der Wiss. zu Leipzig, 60 (1908), p. 94. The present treatment differs in aim from these and has little in common with them.

A very admirable summary of the principal theorems upon groups of points on an algebraic curve is found in Pascal's Repertorium, II[1], pp. 306–355. A good treatise of an elementary sort is Severi's Lezioni di Geometria Algebrica.

An Arithmetic of Points on a Cubic Curve.

2. Any cubic curve, with or without a double point, but not resolvable into curves of lower order, can be reduced by the use of projective transformations alone to the form

$$y^2 = x^3 + \delta_1 x + \delta_2$$

where δ_1 and δ_2 are constants and $\delta_1{}^3/\delta_2{}^2$ is an absolute invariant.

Any polynomial of the form

$$x^3 + \delta_1 x + \delta_2,$$

we shall call $\Delta(x)$, and the cubic curve defined by $y^2 = \Delta(x)$, we shall call a curve K. Any point on a curve K we shall call a point P. Any linear equation in x and y we shall call an equation L, and the line defined by it we shall call a line L. This notation is adopted in order to facilitate a generalization from the case of the cubic curve to that of the hyperelliptic curve, and the same notation will be used in both cases.

3. If a curve K be written in the form $y^2 = \Delta(x)$, then with respect to this curve and this method of representation, a fixed set of finite points on the curve will be said to be in " general position," if no two of the points have the same x-coordinates, i. e., if no two of the points are either coincident or are obtainable one from the other by a mere reflexion in the x-axis. A variable set of points is said to be in " general position," if, except for a zero dimensional set of cases, every particular position of the variable set is in general position according to the previous definition.

We shall have occasion to use the following theorems:

(*a*) Any two finite P's in general position on K lie on one and only one L. This L intersects K again in a uniquely determined P.

(*b*) There is at least one P on K, such that an L tangent to K at one of these P's, has there contact of the second order, in other words, does not meet K at any other distinct point. Such P's are called " Inflexional."

(*c*) If K be non-degenerate and fixed, and F' and F'' be any two curves, degenerate or not; fixed or variable, but of the same degree as K and such

that the finite points of intersection of F' with K consist of a set S together with a single P, e. g., P_1, and the finite points of intersection of F'' with K consist of the same set S together with a single P, e. g., P_2, while S is in general position on K; then P_1 coincides with P_2. We here suppose that the common points of intersection of any two of the curves K, F', and F'' shall be at most a zero-dimensional set of points.

(*d*) The P which lies at infinity we shall call P_0. If an L passes through P_0 and through a finite point on K, the equation of L is satisfied by the line which is drawn through the finite point and parallel to the y-axis. So that any L through P_0 meeting K in a finite P, meets it also in a second P obtainable from the first by a reflexion in the x-axis. Such a reflexion carries an inflexional P into an inflexional P.

Property (*c*) states what is often put in the form, "The K's through eight P's pass also through a ninth," and (*d*) includes a special case of the theorem that an L through two inflexional P's passes through a third.

We shall later prove each of these properties (*a*), (*b*), (*c*), (*d*).

4. If three P's, e. g., P_α, P_β, P_γ, of K are on a single L, we shall indicate this relation by the symbolic equation

$$P_\alpha + P_\beta + P_\gamma = 0.$$

The P obtained from P_α by reflexion in the x-axis, we shall indicate by $-P_\alpha$ or $P_{-\alpha}$, so that we may write

$$P_\alpha + (-P_\alpha) + P_0 = 0.$$

If a P_γ and a $P_{\gamma'}$ are so related that one is the negative of the other, and if furthermore,

$$P_\alpha + P_\beta + P_{\gamma'} = 0,$$

then we shall indicate the relation of P_γ to P_α and P_β by the symbolic equation

$$P_\gamma = P_\alpha + P_\beta,$$

which yields upon reflexion in the x-axis,

$$-P_\gamma = (-P_\alpha) + (-P_\beta).$$

Furthermore,

$$P_\alpha = P_\alpha + P_0,$$

and in particular

$$P_0 = P_0 + P_0.$$

We have thus defined a sort of addition which is obviously commutative, we shall now show that it is also associative.

5. Let us take a particular nondegenerate curve K. We will take on K, three arbitrary P's, e. g., P_α, P_β, P_γ. Through P_α and P_β, there passes

an L, which we shall call L_1', and through P_β and P_γ, another L, viz., L_1''. Let the third P, in which L_1' meets K, be designated by $-P_\delta$, and the third P in which L_1'' meets K, by $-P_\epsilon$. We shall join $-P_\delta$, and $-P_\epsilon$ respectively with P_0, by L_3'' and L_3' respectively. These will determine P_δ and P_ϵ, respectively, as the third P of intersection. Let L_2'' join P_α and P_ϵ, and L_2' join P_γ with P_δ. We shall prove that the third P on L_2'' coincides with the third P on L_2', and this we shall represent by $-P_\zeta$. The incidence relations may be summarized by the following table:

	L_1'	L_2'	L_3'
L_1''	P_β	P_γ	$-P_\epsilon$
L_2''	P_α	$-P_\zeta$	P_ϵ
L_3''	$-P_\delta$	P_δ	P_0

We may suppose P_α, P_β, P_γ, P_δ, P_ϵ, $-P_\delta$, $-P_\epsilon$ to be finite and in general position on K. Now L_1', L_2', L_3' taken together constitute a degenerate curve F', and L_1'', L_2'', L_3'' constitute another degenerate curve F''. Hence if we define as $-P_\zeta$, the P which is the remaining intersection of L_2' with K, then by property (c) mentioned above, L_2'' also must intersect K in this same $-P_\zeta$. Thus the above incidence table is justified. It holds also for special positions, if the L's exist.

In terms of the symbolic equations, we shall therefore have

$$\begin{aligned} P_\delta &= P_\alpha + P_\beta, \\ P_\epsilon &= P_\beta + P_\gamma, \\ P_\zeta &= P_\alpha + P_\epsilon, \\ P_\zeta &= P_\delta + P_\gamma. \end{aligned}$$

$$\therefore \quad P_\alpha + (P_\beta + P_\gamma) = (P_\alpha + P_\beta) + P_\gamma.$$

Hence this addition is associative.

6. Let us now take an arbitrary fixed P_α and any P_ξ. Whenever $P_\xi + P_\alpha$ exists, we may define $P_{\xi+\alpha}$ by the equation

$$P_{\xi+\alpha} = P_\xi + P_\alpha.$$

Similarly $P_{\xi+2\alpha} = P_{\xi+\alpha} + P_\alpha$, and in general $P_{\xi+n\alpha} = P_{\xi+(n-1)\alpha} + P_\alpha$, where n is any positive integer. If now for a particular ξ and n, we have

$$P_{\xi+n\alpha} = P_\xi,$$

then by adding $-P_\xi$ to each member we obtain

$$P_{n\alpha} = P_0,$$

which is independent of the P_ξ chosen originally. Hence, we have the so-called

Theorem of Closure. *If for a given fixed* P_a *and a given constant integer* n, *and for a single choice of* P_ξ, *we should obtain* $P_{\xi+na} = P_\xi$, *then the same is true for every choice of* P_ξ.

Now P_{na} may be determined algebraically from P_a, and since the determination is unique, P_{na} is obtainable rationally from P_a. The condition imposed upon P_a in order that $P_{na} = P_0$ is algebraic, and is known as the " Condition of Closure of the nth Order."

Two-Two Correspondences.

7. Suppose there be given any two-to-two algebraic correspondence between two one-dimensional projective forms.

For example, consider the correspondence between the points of a nondegenerate point conic, and the lines of a non-degenerate line-conic lying in the same plane, when we define the correspondence as being between incident points and lines. For any point on the first conic, we have two lines through this point and tangent to the second conic, and for any line of the second conic, we have two points on this line, and lying also on the first conic. This is of course the case of the Poncelet Polygons.

Any two-two algebraic correspondence between x and M/N, can be written in the form

$$\mathrm{A}(x)\mathrm{M}^2 + 2\mathrm{B}(x)\,\mathrm{M}\mathrm{N} + \Gamma(x)\,\mathrm{N}^2 = 0,$$

when A, B, and Γ are polynomials in x of the second degree *.

8. Let us now start with a correspondence of the form

$$\mathrm{A}\,\mathrm{M}^2 + 2\mathrm{B}\,\mathrm{M}\mathrm{N} + \Gamma\mathrm{N}^2 = 0. \tag{1}$$

Most of the important properties of this correspondence do not depend primarily upon the exact numerical values of the coefficients, but, on the other hand, are unaltered when we replace (1), by a correspondence obtained from it by either or both of the following two transformations:

(a) Replacing M and N by two linearly independent homogeneous linear combinations of them.

(b) Replacing x by a non-singular linear fractional function of x. For the sake of definiteness we may use the freedom at our disposal to reduce (1), to a certain normal form. The discriminant of (1), regarded as a quadratic in M/N, is $\mathrm{B}^2 - \mathrm{A}\Gamma$, which is invariant under (a). It is a polynomial in x, and by means of (b), we may reduce the leading coefficient to zero, the next to unity and the one following to zero. Thus we obtain an expression of the form which we have already called $\Delta(x)$. We shall

* In the elliptic case M and N are independent of x, but in the hyperelliptic case they are polynomials in x, as explained in § 23.

hereafter write $\Delta \equiv B^2 - A\Gamma$. We still have three parameters at our disposal by using (a), and these we may choose in such a way as to reduce the degree of A by one, and the degree of B by two. We may at the same time divide the whole equation (1) by a constant so as to secure that the now leading coefficient of A is unity, and hence as may be seen from the form of Δ, that the leading coefficient of Γ is minus unity, while the second coefficient of Γ becomes equal to the at present second coefficient of A. There are special cases in which this reduction is not possible in just the form stated, as for instance when Δ or A has a pair of coincident roots. We shall however for the present exclude such cases. All the significant coefficients of A, B, Γ, Δ, when these polynomials are expressed in descending powers of x, may be indicated by the following table:

A;	$0,$	$1,$	α_1	
B;	$0,$	$0,$	β_1	
Γ;	$-1,$	$\alpha_1,$	γ_1	
Δ;	$0,$	$1,$	$0,$	$\delta_1.$

Thus we have imposed conditions upon precisely six of the coefficients in the correspondence (1), and have done so in such a manner that $\Delta \equiv B^2 - A\Gamma$ assumes as leading coefficients the numbers indicated in the above scheme.

9. We may write (1) in the irrational form

$$A\frac{M}{N} + B = \pm\sqrt{\Delta}. \tag{2}$$

This suggests that we consider the following two curves or equations,

(a) The fixed K, whose equation is $y^2 = \Delta(x)$,

(b) The variable L, whose equation is $L = 0$, where $L \equiv AM + BN - Ny$, is a rational integral function of x and y.

The variable L meets K in the variable points whose x-coördinates are the roots of (1), considered as an equation in x. But there are also certain fixed points of intersection. For when we eliminate y between the equations for K and for L, every solution of the equation in x so obtained, must be the x-coördinate of a point of intersection of K and L. To eliminate y between the equations for K and L is equivalent to rationalizing (2). When we rationalize (2), we at first obtain, not (1), but

$$A^2M^2 + 2ABMN + B^2N^2 - \Delta N^2 = 0,$$

which differs from (1), by containing throughout the left-hand member, the factor A. When $A = 0$, the equation for the curve K, $y^2 = \Delta$ $(\equiv B^2 - A\Gamma)$ gives $y = +B$ or $-B$, whereas the equation for L gives

$y = \text{B}$. Thus L and K intersect in a fixed P, for which $\text{A} = 0$, $y = \text{B}$, in addition to the variable points of intersection already mentioned.

Not only is this fixed P of intersection determined by A and B, but conversely we may choose a finite P arbitrarily on K and find a unique A and B so as to obtain a correspondence given in the normal form of (1). Indeed from the degree and form of the normalized A and B, respectively, we see that these expressions are uniquely determined by the condition that for a given P, $\text{A} = 0$, $y = \text{B}$. The A and B so determined are such that $\Delta - \text{B}^2$ is divisible by A, for $\Delta - \text{B}^2$ must vanish for $\text{A} = 0$, since this P is also on $y^2 = \Delta$. Hence we may obtain a unique Γ such that $\Delta \equiv \text{B}^2 - \text{A}\Gamma$. With these expressions for A, B, Γ, the correspondence (1) will be in the normal form, and will have the given discriminant Δ.

10. Suppose we be given the curve K and also P_α and P_β selected arbitrarily upon it. Let us consider the rational algebraic process by which $P_{\alpha+\beta}$ may be determined from P_α and P_β.

In place of P_α we may consider the pair of expressions $(\text{A}_\alpha, \text{B}_\alpha)$ such that for P_α, $\text{A}_\alpha = 0$, $y = \text{B}_\alpha$, and similarly we shall replace P_β in our work by $(\text{A}_\beta, \text{B}_\beta)$. The polynomial L in x and y, which vanishes for P_α and for P_β, may be written in either of the following ways:

$$\text{A}_\alpha\text{M} + \text{B}_\alpha\text{N} - \text{N}y, \quad \text{or} \quad \text{A}_\beta\text{M}' + \text{B}_\beta\text{N} - \text{N}y,$$

where N may obviously be taken as the same in the two cases, since the two polynomials, considered as expressions in x and y, represent the same L, so that the coefficient of y in each case must be the same. We are thus led to an identity

$$\text{A}_\alpha\text{M} + \text{B}_\alpha\text{N} \equiv \text{A}_\beta\text{M}' + \text{B}_\beta\text{N},$$

in which A_α, B_α, A_β, B_β are known. A comparison of degrees shows that in general M, M$'$, and N are uniquely determined by this identity except for a constant factor ρ' of proportionality. Now this same L meets K again in what we have defined as $-P_{\alpha+\beta}$ or $P_{-(\alpha+\beta)}$. If $P_{\alpha+\beta}$ be defined by $\text{A}_{\alpha+\beta} = 0$, $y = \text{B}_{\alpha+\beta}$, then $-P_{\alpha+\beta}$ will be defined by using $(\text{A}_{\alpha+\beta}, -\text{B}_{\alpha+\beta})$. This same L may therefore be expressed by

$$\text{A}_{\alpha+\beta}\text{M}'' + (-\text{B}_{\alpha+\beta})\text{N} - y\text{N}.$$

If $\text{A}_{\alpha+\beta}$ were known, we could then solve for $-\text{B}_{\alpha+\beta}$, and incidentally also for M$''$, by using the identity

$$\text{A}_\alpha\text{M} + \text{B}_\alpha\text{N} \equiv \text{A}_{\alpha+\beta}\text{M}'' - \text{B}_{\alpha+\beta}\text{N}.$$

But P_β, and $-P_{\alpha+\beta}$ are determined so far as their x-coördinates are concerned by a correspondence of the form (1). In fact, for the above determined M and N,

$$\text{A}_\alpha\text{M}^2 + 2\text{B}_\alpha\text{MN} + \Gamma_\alpha\text{N}^2$$

vanishes for $A_\beta = 0$ and for $A_{\alpha+\beta} = 0$. Upon comparison of degrees, we see that

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\beta A_{\alpha+\beta},$$

where ρ is a constant factor of proportionality. This serves to determine $A_{\alpha+\beta}$ which was all that was left to find. We shall suppose M and N so selected that $\rho^2 = 1$.

In brief, we obtain $A_{\alpha+\beta}$, and $B_{\alpha+\beta}$ from A_α, B_α, A_β, B_β, by the solution of a series of linear equations obtained by equating coefficients in the identities:

$$\begin{cases} A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\beta A_{\alpha+\beta}, \\ A_\alpha M + B_\alpha N \equiv A_\beta M' + B_\beta N \equiv A_{\alpha+\beta} M'' - B_{\alpha+\beta} N. \end{cases}$$

Explicit relations will appear in a more detailed form in § 12, and §§ 23–24.

If P_β coincides with P_α, the above method may be slightly simplified. In this case A_β is identical with A_α, so that $A_{2\alpha}$ is given directly, and incidentally also M and N, by

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\alpha A_{2\alpha},$$

and $B_{2\alpha}$, by

$$A_\alpha M + B_\alpha N \equiv A_{2\alpha} M'' - B_{2\alpha} N.$$

A discussion of the cases in which this method may break down or be specialized owing to the vanishing of the determinant of the linear equations, will be made later. This applies to the following section also.

11. We shall now consider a shorter method of obtaining $A_{2\alpha}$ and $B_{2\alpha}$ than that just given. Let us determine a polynomial Λ in x such that

$$A_\alpha \Lambda + 2B_\alpha M + \Gamma_\alpha N \equiv 0,$$

which for a given A_α, B_α, Γ_α serves partially to determine Λ, M, N. We may completely determine them by requiring that the coefficient of the highest power of x in $M^2 - \Lambda N$ shall be equal to the above-mentioned ρ. The above bilinear form suggests two quadratic forms, whose discriminants are, respectively, $B_\alpha^2 - A_\alpha\Gamma_\alpha$ and $M^2 - \Lambda N$. The first of these is the Δ which we have used to define K, the second we shall now find to be simply $\rho A_{2\alpha}$. Indeed the identity

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\alpha A_{2\alpha},$$

is found to be satisfied when $M^2 - \Lambda N$ is written instead of $\rho A_{2\alpha}$, for we have

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv A_\alpha M^2 - A_\alpha \Lambda N,$$

since $A_\alpha \Lambda \equiv -2B_\alpha M - \Gamma_\alpha N$. Hence for $\rho^2 = 1$,

$$A_{2\alpha} = \rho(M^2 - \Lambda N).$$

To obtain B_{2a}, we must solve an identity of the form

$$A_a M + B_a N \equiv A_{2a} M'' - B_{2a} N,$$

that is,

$$A_a M + B_a N \equiv \rho(M^2 M'' - \Lambda N M'') - B_{2a} N,$$
$$\equiv \rho(MM'' - N\Pi)M - (B_{2a} + \rho\Lambda M'' - \rho M\Pi)N$$

where Π is arbitrary. If now we choose a Π and an M'' so that

$$\rho(MM'' - N\Pi) \equiv A_a,$$

and that the coefficient of the highest power of x in $\rho(\Lambda M'' - M\Pi)$ is zero, the M'' and Π will be fully determined. We shall then have

$$A_a M + B_a N \equiv A_a M - (B_{2a} + \rho\Lambda M'' - \rho M\Pi)N,$$

whence

$$B_{2a} \equiv -(B_a + \rho\Lambda M'' - \rho M\Pi).$$

This method is a little more expeditious than the general method of determining A_{2a} and B_{2a} given in the preceding section owing to the fact that the polynomials whose coefficients we must equate are here of lower degree than by the method of the preceding section.

Explicit Conditions of Closure.

12. It will be more convenient for some purposes to use the following notation, which differs slightly from that used hitherto in this paper. On writing $-x_n$ for α_1 in the expression for A_{na} (cf. § 8), y_n for β_1, and therefore $-(x_n^2 + \delta_1)$ for γ_1, we obtain

$$A_{na} \equiv x - x_n,$$

$$B_{na} \equiv y_n,$$

$$\Gamma_{na} \equiv -(x^2 + x_n x + x_n^2 + \delta_{1n}),$$

$$\Delta \equiv x^3 + \delta_1 x + \delta_2,$$

where $y_n = \pm\sqrt{x_n^3 + \delta_1 x_n + \delta_2}$. An equation $L = 0$, we may write in the general form:

$$y = l_0 x + l_1,$$

except in the cases when $N = 0$, in which cases we have

$$l_0 x + l_1 = 0.$$

If an L passes through P_{ma} and P_{na}, it will pass also through $-P_{(m+n)a}$, while if one passes through P_{ma} and P_{-na}, it will pass also through $-P_{(m-n)a}$, and by reflexion in the x-axis we obtain two other combinations. If we

be given only A_{ma} and A_{na}, it will be impossible from these data to determine whether P_{ma} or $-P_{ma}$, and P_{na} or $-P_{na}$ is intended. In fact A_{ma} and A_{na} determine $A_{(m+n)a}$ and $A_{(m-n)a}$ by a quadratic construction. In other words, if we be given the x-coördinates only of two of the points of intersection of an L with K, the x-cöordinate of the remaining point of intersection is determined only as being one of a pair of uniquely determined points. We exclude for the moment the special case when the two given points have the same x-coördinate.

Eliminating y between

$$y^2 = x^3 + \delta_1 x + \delta_2, \qquad \text{and} \qquad y = l_0 x + l_1,$$

we obtain

$$x^3 - l_0^2 x^2 + (\delta_1 - 2l_0 l_1)x + (\delta_2 - l_1^2) = 0,$$

as the equation satisfied by the x-coördinates of the points of intersection. If we call these coördinates, x_m, x_n, and x' respectively we have the relations

$$x_m + x_n + x' = l_0^2,$$

$$x_m x_n + x_n x' + x' x_m - \delta_1 = -2l_0 l_1,$$

$$x_m x_n x' + \delta_2 = l_1^2.$$

Eliminating the l's, we have

$$(3) \quad (x_m x_n + x_n x' + x' x_m - \delta_1)^2 - 4(x_m + x_n + x')(x_m x_n x' + \delta_2) = 0.$$

This is a quadratic equation in x', whose roots we have called x_{m+n} and x_{m-n}. We may replace x' in this relation by $x'' + x_l$, where x_l is arbitrary, and obtain

$$(x_m x_n + x_n x'' + x_n x_l + x'' x_m + x_l x_m - \delta_1)^2$$
$$- 4(x_m + x_n + x'' + x_l)(x_m x_n x'' + x_m x_n x_l + \delta_2) = 0,$$

as a quadratic whose roots are $x_{m+n} - x_l$ and $x_{m-n} - x_l$. Expanding, we obtain

$$x''^2[(x_m - x_n)^2]$$
$$+ 2x''\,[(x_n + x_m)(x_m x_n + x_n x_l + x_l x_m - \delta_1)$$
$$- 2(x_m + x_n + x_l)x_m x_n - 2(x_m x_n x_l + \delta_2)]$$
$$+ [(x_m x_n + x_n x_l + x_l x_m - \delta_1)^2 - 4(x_m + x_n + x_l)(x_m x_n x_l + \delta_2)] = 0.$$

By taking the product of the roots we have

$$(x_m - x_n)^2(x_{m+n} - x_l)(x_{m-n} - x_l) = (x_m x_n + x_n x_l + x_l x_m - \delta_1)^2$$
$$- 4(x_m + x_n + x_l)(x_m x_n x_l + \delta_2).$$

By symmetry we have the same right-hand member for $(x_m - x_l)^2 (x_{m+l} - x_n)(x_{m-l} - x_n)$. Hence we have the relation

$$(x_m - x_n)^2(x_{m+n} - x_l)(x_{m-n} - x_l) = (x_m - x_l)^2(x_{m+l} - x_n)(x_{m-l} - x_n)$$

which holds for all choices of x_m, x_n, and x_l.

Let us now replace m and n in the above relation by ml and nl, and introduce expressions C having the following relation to the x's:

$$x_{ml} - x_l \equiv -\frac{C_{m+1}C_{m-1}}{C_m{}^2C_1{}^2}.$$

The above formula now becomes

$$\left[\frac{C_{m+1}C_{m-1}}{C_m{}^2C_1{}^2} - \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2}\right]^2\left[\frac{C_{m+n+1}C_{m+n-1}}{C^2_{m+n}C_1{}^2}\right]\left[\frac{C_{m-n+1}C_{m-n-1}}{C^2_{m-n}C_1{}^2}\right]$$
$$= \left[\frac{C_{m+1}C_{m-1}}{C_m{}^2C_1{}^2}\right]^2 \cdot \left[\frac{C_{m+2}C_m}{C^2_{m+1}C_1{}^2} - \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2}\right] \cdot \left[\frac{C_mC_{m-2}}{C^2_{m-1}C_1{}^2} - \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2}\right].$$

This may be written in the form:

$$\left[\frac{C_1{}^2C_{m+1+n}C_{m+1-n}}{C_{m+2}C_mC_n{}^2 - C_{n+1}C_{n-1}C^2_{m+1}}\right]\left[\frac{C_1{}^2C_{m-1+n}C_{m-1-n}}{C_mC_{m-2}C_n{}^2 - C_{n+1}C_{n-1}C^2_{m-1}}\right]$$
$$= \left[\frac{C_1{}^2C_{m+n}C_{m-n}}{C_{m+1}C_{m-1}C_n{}^2 - C_{n+1}C_{n-1}C_m{}^2}\right]^2.$$

We shall now define C_0 as equal to zero and C_{-m} as equal to $-C_m$. Then, for n arbitrary, and $m = 0$, or $m = 1$, we have identically

$$\frac{C_1{}^2C_{m+n}C_{m-n}}{C_{m+1}C_{m-1}C_n{}^2 - C_{n+1}C_{n-1}C_m{}^2} = 1.$$

But the preceding equation shows that if this equation holds for $m - 1$, and for m, then it must also continue to be true for $m + 1$. Therefore by mathematical induction, the C's are such that for all integer values of m and n

$$C_1{}^2C_{m+n}C_{m-n} = C_{m+1}C_{m-1}C_n{}^2 - C_{n+1}C_{n-1}C_m{}^2, \tag{4}$$

or, as we may write it,

$$\frac{C_{m+n}C_{m-n}}{C_m{}^2C_n{}^2} + \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2} + \frac{C_{1+m}C_{1-m}}{C_1{}^2C_m{}^2} = 0. \tag{4'}$$

13. If we use the abbreviation $F_{m,\,n}$ for the fraction $C_{m+n}C_{m-n}/C_m{}^2C_n{}^2$, by we may derive immediately from (4′), that for any n_1, n_2 and n_3,

$$F_{n_1,\,n_2} + F_{n_2,\,n_3} + F_{n_3,\,n_1} = 0.$$

From the fact that by algebra we have directly

$$(F_{n_1,m} - F_{n_2,m})(F_{n_3,m} - F_{n_4,m}) + (F_{n_2,m} - F_{n_3,m})(F_{n_1,m} - F_{n_4,m}) + (F_{n_3,m} - F_{n_1,m})(F_{n_2,m} - F_{n_4,m}) \equiv 0,$$

we conclude that

$$F_{n_1,n_2} \cdot F_{n_3,n_4} + F_{n_2,n_3} \cdot F_{n_1,n_4} + F_{n_3,n_1} \cdot F_{n\ ,n_4} = 0,$$

which gives in terms of the C's

$$C_{n_1+n_2}C_{n_1-n_2}C_{n_3+n_4}C_{n_3-n_4} + C_{n_2+n_3}C_{n_2-n_3}C_{n_1+n_4}C_{n_1-n_4} + C_{n_3+n_1}C_{n_3-n_1}C_{n_2+n_4}C_{n_2-n_4} = 0.$$

Identities such as these serve frequently to greatly abridge the work of determining a particular C in terms of other C's. We shall now give however a general method of determining each C in terms of the preceding C's. In fact we merely need to write, in (4), first, $n + 1$ in place of m, and obtain

$$C_{2n+1} = \frac{1}{C_1^3}[C_{n+2}C_n^3 - C_{n-1}C_{n+1}^3], \tag{5}$$

and secondly, $n + 1$ in place of m, and $n - 1$ in place of n, and obtain

$$C_{2n} = \frac{C_n}{C_1^2C_2}[C_{n+2}C_{n-1}^2 - C_{n-2}C_{n+1}^2]. \tag{5'}$$

These formulæ (5) and (5′) serve to determine every C_n, if C_1, C_2, C_3, C_4, be known. Furthermore if C_2, C_3, and C_4, be divisible by C_1, formulæ (5) and (5′) serve to establish by induction that every suceeding C is also divisible by C_1. Also, if C_4 is divisible by C_2, then for n odd, or even, every C_{2n} must be divisible by C_2. For suppose this to be the case for $n \leqq \nu + 2$. If n be odd, C_{n-1}^2 and C_{n+1}^2 will both contain C_2^2 as a factor so that from formula (5′), C_{2n} will contain C_2 exactly. If n be even, C_n as well as both C_{n+2}, and C_{n-2} will contain C_2, so that again C_{2n} contains C_2 as a factor. Hence, *any choice of* C_1, C_2, C_3, *and* C_4, *for which* C_1, C_2/C_1, C_3/C_1, *and* $C_4/(C_1 \times C_2)$, *are all polynomials in* x_l, y_l, δ_1, δ_2, *will be such that every* C_n *will be a polynomial in these same quantities*, x_l, y_l, δ_1, δ_2.

14. We may notice that in (4) not only are the terms homogeneous, each being of the same (fourth) degree in the C's, but there is also a homogeneity of another kind, namely with respect to the *square* of the subscripts. Indeed

$$2 \cdot 1^2 + (m + n)^2 + (m - n)^2 = (m + 1)^2 + (m - 1)^2 + 2n^2 = (n + 1)^2 + (n - 1)^2 + 2m^2 = 2(1^2 + m^2 + n^2),$$

which we may call the "quadratic weight" of the expressions involved. Furthermore any expression derived from (4) will be homogeneous in both senses. It will be possible to form from C_1, C_2 and C_n, an expression of degree zero and quadratic weight zero, which shall reduce to unity for $n = 1$, and $n = 2$, and vanish for $n = 0$. Indeed we may write

$$R_n \equiv \frac{C_n \cdot C_1^{(n^2-4)/3}}{C_2^{(n^2-1)/3}},$$

where R_n is a function of the form desired. For every identical relation among the C's we have an analogous relation among the R's, obtainable by merely replacing C_n by R_n. In particular we have formulæ analogous to (5) and (5′), where in the present case we may replace R_1 and R_2 by their common value, unity. Thus we have

$$(6) \qquad R_{2n+1} = R_{n+2}R_n^3 - R_{n-1}R_{n+1}^3,$$

$$(6') \qquad R_{2n} = R_n(R_{n+2}R_{n-1}^2 - R_{n-2}R_{n+1}^2).$$

It will be found that R_3 occurs throughout the entire series of expressions R_n only in the form R_3^3, except in those R_n's for which n is a multiple of three, in which case R_3 occurs also as a simple factor of the entire expression, as is readily proved by induction.

15. We may write U for R_3^3 and V for R_4. We then have directly from (6) and (6′),

$$R_0 = 0,$$

$$R_1 = 1,$$

$$R_2 = 1,$$

$$R_3 = U^{1/3},$$

$$R_4 = V,$$

$$R_5 = V - U,$$

$$R_6 = U^{1/3}(V - U - V^2),$$

$$R_7 = (V - U)U - V^3,$$

$$R_8 = V[(V - U)(2U - V) - UV^2],$$

$$R_9 = U^{1/3}[V^3(V - U - V^2) - (V - U)^3],$$

$$R_{10} = (V - U)[V^2(UV - U^2 - V^3) - U(V - U - V^2)^2],$$

$$R_{11} = (UV - U^2 - V^3)(V - U)^3 - UV(V - U - V^2)^3,$$

. . . .

$$R_{-n} = - R_n.$$

16. Thus far, the C's and the R's have depended upon an arbitrary integer l. Putting $l = 1$, we have

$$x_m - x_1 = -\frac{C_{m+1}C_{m-1}}{C_m^2 C_1^2},$$

and more generally from (4), we have

$$x_m - x_n = -\frac{C_{m+n}C_{m-n}}{C_m^2 C_n^2}.$$

Now x_0 is the x at infinity so that $x_m - x_n$ becomes infinite when and only when either x_m or x_n coincides with x_0. Furthermore $x_m - x_n$ vanishes when and only when either x_{m+n} or x_{m-n} coincides with x_0. Thus the vanishing of C_m is a necessary and sufficient condition that x_m coincides with x_0, provided only that the C's be integral functions of x_1 and y_1. This may be readily secured. We have merely to take C_1 equal to unity, and C_2 equal to $-2y_1$, although other choices are also possible. We will find immediately that C_3 then assumes an integral form, and as we shall show in the following section, § 17, C_4 is then integral and divisible by C_2. The discussion in § 13 shows that all the C's are integral functions of x_1 and y_1.

The identity $\mathrm{A}_a\Lambda + 2\mathrm{B}_a\mathrm{M} + \Gamma_a\mathrm{N} \equiv 0$, admits, when ρ is taken as minus one, the solutions*

$$\Lambda \equiv x + 2x_1,$$

$$\mathrm{M} \equiv \frac{1}{2y_1}(3x_1^2 + \delta_1),$$

$$\mathrm{N} \equiv 1.$$

The identity

$$\rho(\mathrm{MM}'' - \mathrm{N}\Pi) \equiv \mathrm{A}_a,$$

yields, cf. § 11

$$\mathrm{M}'' \equiv \frac{1}{2y_1}(3x_1^2 + \delta_1),$$

$$\Pi \equiv (x - x_1) + \frac{1}{4y_1^2}(3x_1^2 + \delta_1)^2.$$

The equation, $\rho(\mathrm{M}^2 - \Lambda\mathrm{N}) = A_{2a}$, gives

$$A_{2a} = x + 2x_1 - \frac{1}{4y_1^2}(3x_1^2 + \delta_1)^2,$$

$$= x - \frac{1}{4y_1^2}(x_1^4 - 2\delta_1 x_1^2 - 8\delta_2 x_1 + \delta_1^2).$$

* It may be seen that we must throughout suppose that every N is taken as unity, that ρ is always minus unity, and M, M′, and M″ for a given L always coincide, in order that the normalizing conditions which we have imposed be satisfied; but cf. §§ 23–24.

Similarly the equation $B_{2\alpha} = -(B_\alpha + \rho\Lambda M'' - \rho M\Pi)$ gives

$$B_{2\alpha} = \frac{1}{8y_1^3}[x_1^6 + 5\delta_1 x_1^4 + 20\delta_2 x_1^2 - 5\delta_1^2 x_1^2 - 4\delta_1\delta_2 x_1 - (8\delta_2^2 + \delta_1^3)].$$

Since $A_{2\alpha} = x - x_2$, we may compute C_3 from

$$x_2 - x_1 = -\frac{C_3 C_1}{C_2^2 C_1^2}.$$

In fact

$$C_3 = 3x_1^4 + 6\delta_1 x_1^2 + 12\delta_2 x_1 - \delta_1^2.$$

We shall now determine C_4.

17. From the quadratic (3), whose roots are x_{m+n} and x_{m-n}, we obtain for the sum of the roots:

$$x_{m+n} + x_{m-n} = \frac{2}{(x_m - x_n)^2}[(x_m x_n + \delta_1)(x_m + x_n) + 2\delta_2]. \tag{7}$$

Now $(x_{m-n} - x_1)(x_m - x_n)^2$ may be written in the form

$$-\frac{C_{m-n+1}C_{m-n-1}}{C_{m-n}^2 C_1^2} \cdot \frac{C_{m+n}^2 C_{m-n}^2}{C_m^4 C_n^4} = \frac{C_{m-n+1}C_{n-m+1}C_{m+n}^2}{C_m^4 C_n^4}.$$

Thus, although $(x_m - x_n)^2$ vanishes for $n = m$, yet the expression $(x_{m-n} - x_1)(x_m - x_n)^2$ reduces for $n = m$ to C_{2m}/C_m^8. Hence for $n = m$, we obtain from (7)

$$\begin{aligned} C_{2m} &= 2C_m^8[(x_m^2 + \delta_1)2x_m + 2\delta_2], \\ &= 4C_m^8 \cdot y_m^2, \\ &= 4C_m^8 \cdot B_{m\alpha}^2. \end{aligned}$$

The sign of $B_{m\alpha}$ depends upon the sign chosen for B_α, i. e., y_1, and may be found from the identities

$$A_\alpha M + B_\alpha \equiv A_{m\alpha}M + B_{m\alpha} \equiv A_{(m+1)\alpha}M - B_{(m+1)\alpha}.$$

Eliminating M, this gives

$$[A_{(m+1)\alpha} - A_\alpha][B_{m\alpha} - B_\alpha] + [A_{m\alpha} - A_\alpha][B_{(m+1)\alpha} + B_\alpha] \equiv 0,$$

which is satisfied identically for $B_{m\alpha} = -(C_{2m}/2C_m^4)$, and $B_{(m+1)\alpha} = -(C_{2(m+1)}/2C_{m+1}^4)$, but not for any other combinations of signs.

Since, now, we have already found $B_{2\alpha}$, we may determine C_4. Indeed, we have immediately,

$$C_4 = -4y_1[x_1^6 + 5\delta_1 x_1^4 + 20\delta_2 x_1^2 - 5\delta_1^2 x_1^2 - 4\delta_1\delta_2 x_1 - (8\delta_2^2 + \delta_1^3)].$$

Having found C_2, C_3 and C_4, we may either find the R's first, or compute

the C's directly by the recursion formulæ, (5) and (5′). The C's determine a set of "conditions of closure," $C_n = 0$ being a "condition of closure of the nth order."

Thus *every* $\mathrm{C_n}$ *is a polynomial in* $\mathrm{x_1}$; *for* n *odd the coefficients of the various powers of* $\mathrm{x_1}$, *are themselves polynomials in* δ_1 *and* δ_2 *with integral numerical coefficients*; *for* n *even, the same is true except that* $\mathrm{y_1}$ *also occurs as a factor of the entire expression.* We have just seen this to be the case for C_2, C_3 and C_4. Formulæ (5) and (5′) serve immediately to establish by induction the truth of this statement, in all cases.

Generalization to Hyperelliptic Curves.

18. The statements in §§ 3–6, 8–11, inclusive, are capable of a wide extension without even verbal modification. We shall now proceed to present this more general problem.

Let us take an arbitrary positive integer p, and form a polynomial $\Delta(x)$ of degree $2p + 1$, of the form,

$$\Delta(x) \equiv x^{2p+1} + \delta_1 x^{2p-1} + \delta_2 x^{2p-2} + \cdots + \delta_{2p-1}x + \delta_{2p},$$

the δ's being constants. The equation $y^2 = \Delta(x)$ defines a curve K, which is said to be hyperelliptic and of genus p. Furthermore this equation may be considered as a normal form under a certain class of transformations, viz., the birational group, of the most general non-degenerate hyperelliptic curve. We shall suppose that $\Delta(x) = 0$ does not admit multiple roots, as otherwise the curve has finite multiple points, which we shall suppose to have been removed. We shall carry over the definition given in § 3, of "general position," extending it only by defining a set some of whose points are at infinity as being in general position, if the set of the finite points when existent is in general position in the sense of the previous definition. An arbitrary set of p points in general position on K, shall be defined as a set P or simply as a P.

There are certain variable curves L, having for equations, expressions of the form $L(x, y) = 0$, which meet K in $3p$ variable points of intersection, with perhaps one common fixed point at infinity. These curves are such that any $2p$ of the $3p$ variable points of intersection may be assigned at random, but the curve L is thereby completely identified, and the positions of the remaining p intersections are then determinate. We shall consider the form of L, and show that the L's which we obtain satisfy not only the properties (a) and (b), but also (c) and (d), mentioned in § 3 as properties of the cubic. Incidentally we shall be proving the theorems in that case also.

19. The equation of an algebraic curve F, in the (x, y) plane can be written in the form $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y. We may separate $F(x, y)$ into an even and an odd function of y so as to write

$$F(x, y) \equiv F_0(x, y^2) + yF_1(x, y^2).$$

The curve F, in so far as the equation $y^2 = \Delta(x)$ is supposed to be satisfied, i. e., in so far as the points of K alone are considered, is the same as the curve F', whose equation is

$$F'(x, y) \equiv F_0[x, \Delta(x)] + yF_1[x, \Delta(x)].$$

Clearly this is finite for all finite points on the curve K. Moreover, every rational function of x and y, which is finite for all finite points on the curve K, may be written in the form, $F_0(x) + yF_1(x)$.

For suppose we be given any rational function $G(x, y)/H(x, y)$, where G and H are polynomials. If we write H in the form $H_0(x, y^2) + yH_1(x, y^2)$, and then multiply numerator and denominator of the fraction by $H_0 - yH_1$, the denominator will become an even function of y. Replacing y^2 throughout by $\Delta(x)$, we shall have an expression of the form

$$\frac{F_0(x) + yF_1(x)}{H_2(x)}.$$

We wish now to show that if this is to be finite for all finite points on the curve, then the denominator $H_2(x)$ is a factor of both $F_0(x)$ and $F_1(x)$.

If $[F_0(x) + yF_1(x)]/H_2(x)$ is to be finite, the same is true of $[F_0(x) - yF_1(x)]/H_2(x)$, for these differ only in the sign of y, while the curve $y^2 = \Delta(x)$ is symmetric with respect to the x-axis, so that should one become infinite at a point (x, y) the other must become infinite at $(x, -y)$. Both the sum and the product of finite functions are finite so that

$$\frac{2F_0(x)}{H_2(x)} \quad \text{and} \quad \frac{F_0^2(x) - y^2F_1^2(x)}{H_2^2(x)} \quad \text{or} \quad \frac{F_0^2(x) - \Delta(x)F_1^2(x)}{H_2^2(x)},$$

are finite for finite points on K. From the first expression we see that every root of the polynomial $H_2(x)$ must also be a root of the polynomial $F_0(x)$, and hence by the second expression, every simple root of $H_2(x)$ is a double root of ΔF_1^2. But by hypothesis $\Delta(x) = 0$ has no double roots, so that any root of $H_2(x)$, is also a root of $F_1(x)$. Hence every factor of $H_2(x)$ is a factor of both $F_0(x)$ and of $F_1(x)$, and we may divide out $H_2(x)$ entirely, as was to be proved.

20. If either $F_0(x)$ or $F_1(x)$ be identically zero or, more generally, if F_0 and F_1 be not prime to each other, then the intersections of the curve

$F \equiv F_0(x) + yF_1(x) = 0$, with the curve K, will not be in general position, since every common linear factor of F_0 and F_1 determines two points on K, which differ only in the sign of y. Let us suppose F_0 and F_1 prime to each other. The x-coördinates of the points of intersection of $F \equiv F_0(x) + yF_1(x) = 0$, with $y^2 = \Delta(x)$ are given by the roots of the equation

$$F_0^2(x) - \Delta(x)F_1^2(x) = 0.$$

Since $\Delta(x)$ is of *odd* degree, and $F_0^2(x)$ and $F_1^2(x)$ are necessarily of *even* degree, it will be impossible for the leading coefficients of $F_0^2(x)$ and $\Delta(x) \cdot F_1^2(x)$ to cancel each other. Let us suppose $r, \geqq p + 1$, points of intersection chosen arbitrarily and in general position on K. This will impose r linear conditions upon the coefficients of $F \equiv F_0 + yF_1 = 0$. For $r - p$ even, we shall write $r = p + 2s$, and for $r - p$ odd, we shall write $r = p + 2s - 1$, where $s \geqq 1$ in both cases. The highest possible degrees of F_0, and of F_1, for which $F_0^2 - \Delta F_1^2$ shall be of degree $r + p$ are indicated by the following table to be $r - s$ and $s - 1$, respectively.

Terms	*Degree*	*Deg. for* $r - p = 2s - 1$	*Deg. for* $r - p = 2s$
F_0	$r - s$	$p + s - 1$	$p + s$
F_1	$s - 1$	$s - 1$	$s - 1$
F_0^2	$2r - 2s$	$r + p - 1$	$r + p$
ΔF_1^2	$2p + 2s - 1$	$r + p$	$r + p - 1$
$F_0^2 - \Delta F_1^2$	$r + p$	$r + p$	$r + p$

Now the total number of homogeneous coefficients in F_0 is $r - s + 1$, and in F_1 is s, so that if Δ be given and fixed, $F \equiv F_0 + yF_1$ can be subjected to exactly r linearly independent conditions, while, on the other hand, the number of intersections of F with K will be $r + p$. Hence, the curve F which has $r + p$ intersections with K cannot have more than r independent coefficients, it being supposed that $r \geqq p + 1$. The statement also holds for $r = p$ since $F_0^2 - \Delta F_1^2$ can be of degree $2p$ only when F_1 vanishes identically, Δ being of degree $2p + 1$, since, as we have remarked, the leading coefficients of the two terms can never cancel each other. But in this case, F_0 must be of degree p exactly, and the equation $F_0^2 = 0$ contains but p independent coefficients.

21. We may choose as the curves L, the curves whose equations are of the form $L(x, y) \equiv E(x) - y\mathrm{N}(x) = 0$, where E and $-\mathrm{N}$ are polynomials of degree $2p - q$ and $q - 1$, respectively, q being $\frac{1}{2}p$ when p is even, and $\frac{1}{2}(p + 1)$ when p is odd. Thus E and $-N$ are cases of F_0 and F_1, for which $r = 2p$ and $s = q$. Hence if $2p$ points, i. e., two P's, be given on K, no two of the points having the same x-coördinate, then there is a unique L through them. Since $E^2 - \Delta\mathrm{N}^2$ is of degree $3p$, it will vanish

for yet a third P the y-coördinate of each point of which is determined linearly by the equation $\mathrm{N}y = E$. Thus property (a) holds true for the general hyperelliptic curve, K.

The condition that the $3p$ points of intersection of an L with K, reduce to p sets of three coincident points each, i. e., that $E^2 - \Delta\mathrm{N}^2$ be a perfect cube, clearly requires that the coefficients of E and N satisfy certain algebraic relations. Indeed if we write

$$E^2 - \Delta\mathrm{N}^2 \equiv Q^3,$$

where Q is polynomial in α of degree p all of whose coefficients are unknowns, we shall have $3p + 1$ equations which are homogeneous in the $2p + 1$ homogeneous coefficients of E and N, and of degree two in these, and also homogeneous and of degree three in the $p + 1$ homogeneous coefficients of Q. The number of solutions will be found to be 3^{2p}. Thus property (b) holds also in this general case.

22. We shall now make the hypothesis required for property (c), as stated in § 3. The curves F' and F'' of § 3, may be represented by equations of the forms $F_0'(x) + yF_1'(x) = 0$ and $F_0''(x) + yF_1''(x) = 0$ respectively, so far as points of K are concerned. Now these two rational functions are known to have r of their total $r + p$ zeros on K in common and in general position on K. But these r zeros which constitute the set S fully determine the rational function, except for a constant factor. Thus $F_0'(x) + yF_1'(x)$ and $F_0''(x) + yF_1''(x)$ differ from each other by at most a constant factor, and have all of their zeros on K in common. This proves (c) to hold true for the hyperelliptic case.

In order that $E^2(x) - \Delta(x)\mathrm{N}^2(x) = 0$, shall have p coincident roots at infinity, it is necessary and sufficient that N vanish identically, while E reduces to degree p. In this case L reduces to p straight lines parallel to the y-axis. In the equations satisfied by an inflexional set of p points, we do not use y explicitly and N enters in these, only in the form N^2. Hence if $E - \mathrm{N}y$ yields one solution, $E + \mathrm{N}y$ will give another. Thus (d) also is valid in the present problem. The entire discussion in §§ 3–6, inclusive, applies to the hyperelliptic case, since it follows from the properties (a), (b), (c), (d).

23. We may now start with the correspondence

$$\text{(1)} \qquad \mathrm{A}(x)\mathrm{M}^2(x) + 2\mathrm{B}(x)\mathrm{M}(x)\mathrm{N}(x) + \Gamma(x)\mathrm{N}^2(x) = 0,$$

where A, B, Γ, M, N, are polynomials in x. We may suppose the degrees of A, B and Γ to be given as equal to $p + 1$. By applying the reductions used in § 8, we find that, finally, if our M and N have been suitably chosen, A, B, Γ, M, N are of the following form.

$$
\begin{aligned}
\mathrm{A} &\equiv \qquad\qquad\quad x^p + \alpha_1 x^{p-1} + \alpha_2 x^{p-2} + \cdots + \alpha_p,\\
\mathrm{B} &\equiv \qquad\qquad\qquad\qquad \beta_1 x^{p-1} + \beta_2 x^{p-2} + \cdots + \beta_p,\\
\Gamma &\equiv -x^{p+1} + \alpha_1 x^p + \gamma_1 x^{p-1} + \gamma_2 x^{p-2} + \cdots + \gamma_p,\\
\mathrm{M} &\equiv \mu_0 x^{p-q} + \mu_1 x^{p-q-1} + \cdots + \mu_{p-q},\\
\mathrm{N} &\equiv \nu_0 x^{q-1} + \nu_1 x^{q-2} + \cdots + \nu_{q-1},
\end{aligned}
$$

and therefore where $\Delta \equiv \mathrm{B}^2 - \mathrm{A}\Gamma$ is of the form

$$\Delta \equiv x^{2p+1} + \delta_1 x^{2p-1} + \delta_2 x^{2p-2} + \cdots + \delta_{2p}.$$

The E that we have previously used may hereafter be considered as having been an abbreviation for AM + BN. We have merely to take the x-coördinates of p of the points of intersection of $E(x) - y\mathrm{N}(x)$ with K, and form the equation $\mathrm{A}(x) = 0$ which has these for roots. The y-coördinates of these same p points may be used to determine a corresponding B by means of the equations

$$y_i = \beta_1 x_i^{p-1} + \beta_2 x_i^{p-2} + \cdots + \beta_p \qquad i = 1, 2, \cdots p,$$

whose determinant is the Vandermonde determinant of the x's. Since the p points (x_i, y_i) are supposed to have their x-coördinates distinct, we may always determine the desired $\mathrm{B}(x)$, and determine it uniquely. But this choice of A and B satisfies the condition that for $\mathrm{A} = 0$, E reduces to NB. If we let the x-coördinates of the remaining points of intersection of $y = \mathrm{B}$ with L, be the roots of a polynomial M put equal to zero, the E will be necessarily of the form, $E \equiv \mathrm{BN} + \mathrm{AM}$ if the arbitrary constant factor in N and M be properly chosen.

The remarks of §§ 8–10, inclusive, are now immeditely applicable to the correspondence (1), extended in this manner.

It may be remarked that when working in the projective plane, i. e., closing the finite plane by means of a line at infinity, the point at infinity on the curve is by no means a simple, but is a p-tuple point. For most purposes, it is convenient to suppose K situated in a "function plane." In other words, we shall in speaking of the point at infinity regard A, B, Γ as *forms*, which do not become infinite at infinity, while, on the other hand, A and B vanish there to the first and second orders, respectively. With this convention, infinity becomes an ordinary branch point for the curve K, and the L's do not in general pass through this point. This matter is not however of great significance in this connection, as we are concerned almost exclusively with those points alone which lie on the curve K, whose internal properties are essentially the same in either plane.

24. In the identity

$$\mathrm{A}\Lambda + 2\mathrm{BM} + \Gamma\mathrm{N} \equiv 0,$$

the term ΓN is of degree $p + q$, which is always higher than the degree $2p - q - 1$ of the term $2BM$. But A is itself of degree p, so that Λ must be of degree q, since its leading term must cancel that of ΓN. So that we must have $\lambda_0 = \nu_0$, where λ_0 is the leading coefficient in Λ. Now Λ, M, N contain altogether $(q + 1) + (p - q + 1) + q = p + q + 2$ homogeneous coefficients and since the degree of the left hand member is $p + q$, we have $p + q + 1$ homogeneous equations to be satisfied. Thus the solution is unique apart from an arbitrary factor. In general we may impose the condition that for $p = 2q - 1$, $\lambda_0 = 1$, and for $p = 2q$, $\mu_0 = 1$. By this means we shall secure that the leading coefficient in $M^2 - \Lambda N$ is equal to $\rho = (-1)^p$. The Λ, M, N will now be fully determined if the A, B, Γ are supposed given.

In the identity

$$\rho(MM'' - N\Pi) \equiv A$$

where M'' is of degree $p - q$, and Π of degree q, the coefficient of x^p in the left-hand member is, for $p = 2q - 1$, π_0, and for $p = 2q$, μ_0''. And since this is to be an identity, we must have for $p = 2q - 1$, $\pi_0 = 1$, and for $p = 2q$, $\mu_0'' = 1$. The degree of the left hand member is p, while the number of coefficients in M'' and Π together is $p + 2$. Since the equations which are obtained by equating coefficients are non-homogeneous and only $p + 1$ in number, we may in general impose an additional linear condition. We shall require for $p = 2q - 1$, $\mu_0'' = \mu_0$, and for $p = 2q$, $\pi_0 = \lambda_0$. With this restriction, the M'' and Π are completely determined, and in such a manner that $\lambda_0\mu_0'' - \pi_0\mu_0 = 0$, for p odd or even, so that in either case the coefficient of x^p in $\rho(\Lambda M'' - \Pi M)$ is zero. The discussion in § 11, now applies without modification to the hyperelliptic case.

It is important to note that in the above nonhomogeneous system the determinant of the coefficients of the system is for $p = 2q - 1$ and also for $p = 2q$, precisely the resultant of M and N. But if M and N have a common factor, the L whose equation is $AM + BN - Ny = 0$, and which has contact at the P defined by the (A, B) in question, will pass through the point at infinity, and will contain one or more lines parallel to the y-axis. But these lines can have contact with the curve only at the branch points. Thus the above method breaks down only when the given set P has one or more points at the branch points. But the situation in this special case is too obvious to require further discussion.

The General Algebraic Curve.

25. On any algebraic curve of genus p, we may consider an arbitrary series of groups g_{3p}^{2p} of points, any group of which contains $3p$ points and is determined by $2p$ of them. There will be, of course, in general, certain

fixed points in addition, should we cut the series out by a linear family of adjoints. Since $3p > 2p - 2$, the series cannot be special. There exist certain " inflexional " groups G of p sets of three coincident points each. We may choose one such set as P_0, and then determine with respect to this P_0, an arithmetic of groups of p points on the curve. It is only in the hyperelliptic case, however, that an individual point can be considered as having a negative obtained by a " reflexion." The Riemann-Roch Theorem in its simplest form shows that the property expressed in (c) holds good for the general algebraic curve. Thus there is an analogous theorem of closure for any algebraic curve of genus $p > 0$.

The Transcendental Treatment.

26. The Inversion Problem of Jacobi is concerned with the correspondence between the coördinates of a set of p points on an algebraic curve of genus p and the p sums of p linearly independent integrals of the first kind, the sum in each case being of the values assumed by one of these integrals at each of the p points in question. If we consider a group P to represent the p sums here mentioned, then by the addition of P's is meant only the addition of integrals, so that the associative law no longer needs proof, when once it is proved that the group of points is uniquely determined by the p sums of integrals. The equations now become linear equations, *modulis* the periods.

If in the elliptic case we take $x_1 = \wp(u)$ where $\wp$ is the Weierstrassian function, then x_n is $\wp(nu)$, and C_n is a theta function of u of the nth order most conveniently written in the form

$$C_n = \frac{\sigma(nu)}{\sigma(u)^{n^2}}.$$

The algebraic treatment does not define C_n except for n an integer, and by reversing the processes, for n, a fraction. It may be noted, however, that the C_n's are defined for curves in modular geometries by the present treatment, where transcendental methods cannot be applied.

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